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DYNAMIC PROGRAMMING AND LAGRANGE MULTIPLIERS

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SUMMARY

In this paper it is shown that a combination of the classical Lagrange multiplier formalism and the functional equation technique of dynamic programming enables ~~us to treat~~ a number of types of variational problems involving the computation and tabulation of functions of M variables to be treated by computing first sequences of functions of K variables, and then sequences of functions of ~~M-K~~ variables, where K may be chosen within the range $1 \leq K \leq M-1$. The choice of K depends upon the process.

This reduction in the dimensionality of the functions involved is equivalent to an increase in the capability of modern digital computers as far as dynamic programming processes are concerned.

DYNAMIC PROGRAMMING AND LAGRANGE MULTIPLIERS

1. INTRODUCTION

The purpose of this note is to indicate how a suitable combination of the classical method of the Lagrange multiplier and the functional equation method of the theory of dynamic programming[1] can be used to solve numerically and treat analytically a variety of variational problems that cannot readily be treated by either method alone.

A series of applications of the method presented here will appear in further publications.

2. FUNCTIONAL EQUATION APPROACH

Consider the problem of maximizing the function

$$F(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N} g_i(x_i), \tag{2.1}$$

subject to the constraints

$$\begin{aligned} &(a)\quad \sum_{j=1}^{N} a_{ij}(x_j) \le c_i,\ i = 1,2,\ldots,M, \\ &(b)\quad x_i \ge 0, \end{aligned} \tag{2.2}$$

where the functions $a_{ij}(x)$, $g_i(x)$ are taken to be continuous for $x \ge 0$, and monotone increasing. For $c_i \ge 0$, define the sequence of functions

$$f_N(c_1,c_2,\ldots,c_M) = \max_{\{x\}} F(x_1,x_2,\ldots,x_N)$$

for $N \ge 1$.

Then $f_1(c_1,c_2,\ldots,c_M)$ is determined immediately, and, employing the principle of optimality,[1] we obtain the recurrence relation

$$f_{k+1}(c_1,c_2,\ldots,c_M) = \max_{0\le a_{i,k+1}(x)\le c_i} [g_{k+1}(x)+f_k(c_1-a_{1,k+1}(x),\ldots,c_M-a_{M,k+1}(x))], \tag{2.4}$$

for $k = 1,2,\ldots,N-1$.

Due to the limited memory of present-day digital computers, this method founders on the reef of dimensionality when $M \geq 4$. If we wish to treat applied problems of greater and greater realism, we must develop methods capable of handling problems involving higher dimensions.

In this paper we shall present one method of overcoming these dimensionality difficulties.

3. FUNCTIONAL EQUATIONS AND LAGRANGE MULTIPLIERS

The method of the Lagrange multiplier in classical variational theory consists of forming the function

$$\phi(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N} g_i(x_i) - \sum_{i=1}^{M} \lambda_i\Big(\sum_{j=1}^{N} a_{ij}(x_j)\Big), \tag{3.1}$$

where the λ_i are parameters determined subsequently by means of (2.2), and then utilizing a direct variational approach on this new function.

We shall employ an approach intermediate between this method and the method sketched in §2.

Consider the function

$$F(x_1,x_2,..,x_N;\lambda_1,\lambda_2,\ldots,\lambda_K) = \sum_{i=1}^{N} g_i(x_i) - \sum_{i=1}^{K} \lambda_i\Big(\sum_{j=1}^{N} a_{ij}(x_j)\Big), \tag{3.2}$$

where $1 \leq K \leq M-1$. We wish to maximize this function over the region defined by the constraints

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$$\text{(a)} \quad \sum_{j=1}^{M} a_{ij}(x_j) \leq c_i, \quad i = K+1, \ldots, M,$$

$$\text{(b)} \quad x_i \geq 0. \tag{3.3}$$

For fixed values of the λ_i, we have a problem of precisely the type discussed in §2, with the advantage that we now require functions of degree M-K for a computational solution.

Once the sequence $\{\phi_N(c_{K+1}, \ldots, c_N; \lambda_1, \lambda_2, \ldots, \lambda_K)\}$, $N = 1,2,\ldots,$

$$\phi_N(c_{K+1}, \ldots, c_N; \lambda_1, \lambda_2, \ldots, \lambda_K) = \max_{\{x\}} F(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots, \lambda_K) \tag{3.4}$$

has been computed, we vary the parameters λ_i to determine the range of the parameters $c_1, c_2, \ldots, c_K$.

We have thus partitioned the computation of the original sequence of functions of M variables into the computation of a sequence of functions of K variables, followed by the computation of a sequence of functions of M-K variables. The choice of K will depend upon the process.

There are a number of rigorous details which we will discuss elsewhere.

4. SUCCESSIVE APPROXIMATIONS

Since the Lagrange multipliers are intimately connected with "marginal returns" or "prices," in a number of applied problems we begin with a certain hold on the process as far as approximate solutions are concerned. Iterative techniques based upon this observation, and the connection with optimal search procedures will be discussed elsewhere.

5. APPLICATION TO THE CALCULUS OF VARIATIONS

In the theory of control processes,[3] one encounters a problem such as that of minimizing a non-analytic functional such as

$$J_1(u) = \int_0^T |1-u|dt \tag{5.1}$$

over all functions v(t) satisfying the constraints

$$\begin{aligned} &(a) \quad -k_1 \le v(t) \le k_2, \qquad 0 \le t \le T, \\ &\int_0^T |v(t)|dt \le c_1 \end{aligned} \tag{5.2}$$

where u and v are connected by a relation

$$\frac{du}{dx} = g(u,t,v), \quad u(0) = a. \tag{5.3}$$

Replacing $J_1(u)$ by

$$J_2(u) = \int_0^T |1-u|dt + \lambda \int_0^T |v(t)|dt \tag{5.4}$$

$\lambda \ge 0$, we can employ functions of two variables in determining the analytic or numerical solution, rather than functions of three variables; cf.[2]. This reduction by one in dimensionality results simultaneously in a tremendous saving in computing time, and in a great increase in accuracy of the numerical results.

1. R. Bellman, The Theory of Dynamic Programming, Bull. Amer. Math. Soc., vol. 60(1954), pp 503-516.

2 R. Bellman, Dynamic Programming and a New Formalism in the Calculus of Variations, Proc. Nat. Acad. Sci., 40, 231-235, 1954.

3 R. Bellman, I. Glicksberg, and O. Gross, On Some Variational Problems Occurring in the Theory of Dynamic Programming, Rendiconti del Palermo, Serie II, Tomo III, 1954, 1-35.